On factorisable, not simple groups.

To Professor L. RÉDEI on his 50th birthday.

By J. SZÉP in Szeged.

A group $\mathfrak{G}$ is called factorisable by its subgroups $\mathfrak{H}$ and $\mathfrak{K}$ if each element of $\mathfrak{G}$ may be written in the form $G = HK$ with H in $\mathfrak{H}$, K in $\mathfrak{K}$ (written $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$).

It was G. ZAPPA [9] who has considered first factorisable groups; he has supposed that the two subgroups have no element in common except the unit element. Since that several papers have been published dealing with factorisable groups [1]—[8]. The general case where the factors may have a subgroup $\mathfrak{D} \neq E$ of $\mathfrak{G}$ in common was treated by the author and L. RÉDEI [8]. For the case that $\mathfrak{D}$ is a normal subgroup G. CASADIO [1] has obtained a similar result as G. ZAPPA.

The interest of factorisable groups may be judged from the fact that an infinite number of simple groups are factorisable [6]. Moreover, it is possible to obtain some criteria for the simplicity of factorisable groups [7].

The present paper contains two principal results:

Theorem 1. *If a group $\mathfrak{G}$ of finite order is factorisable by two abelian groups, then $\mathfrak{G}$ is not simple.*

Theorem 2. *If a group $\mathfrak{G}$ of finite order is factorisable by two abelian groups $\mathfrak{H}$ and $\mathfrak{K}$ and the orders of $\mathfrak{H}$ and $\mathfrak{K}$ are relatively prime, then $\mathfrak{G}$ is solvable.*

Theorem 2 is connected with the following theorem of O. HÖLDER:

If $\mathfrak{G}$ is a group of finite order and if all Sylow-groups of $\mathfrak{G}$ are cyclic, then $\mathfrak{G}$ is solvable and factorisable by two cyclic groups with relatively prime orders.

The converse is also true, it follows from the original theorem:

If a group $\mathfrak{G}$ of finite order is factorisable by two cyclic groups whose orders are relatively prime, then $\mathfrak{G}$ is solvable.

Theorem 2 is a generalization of the converse of HÖLDER's theorem.

For the proof of Theorems 1 and 2 it is necessary to recall some facts contained in earlier papers:

Theorem A. [8] *Let the group* $\mathfrak{G}$ *be factorisable by the proper subgroups* $\mathfrak{H}$ *and* $\mathfrak{K}$, *and let* $\mathfrak{H} \cap \mathfrak{K} = \mathfrak{D} \neq E$. *If* $\mathfrak{D}$ *has a proper subgroup* $\mathfrak{D}'$ *which is a normal subgroup of* $\mathfrak{H}$, *then the group* $\mathfrak{G}$ *is not simple.*

Theorem B. [5] *If in the finite group* $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ *the orders of the factors* $\mathfrak{H}$ *and* $\mathfrak{K}$ *are relatively prime, then every normal subgroup* $\overline{\mathfrak{G}}$ *of* $\mathfrak{G}$ *is of the form* $\overline{\mathfrak{G}} = \overline{\mathfrak{H}}\overline{\mathfrak{K}}$ *where* $\overline{\mathfrak{H}}$ *and* $\overline{\mathfrak{K}}$ *are normal subgroups of* $\mathfrak{H}$ *and* $\mathfrak{K}$ *respectively.*

After these preliminaries we turn our attention to the proofs of Theorems 1 and 2.

Proof of Theorem 1. If $\mathfrak{D} \neq E$ then the statement is evident. If $\mathfrak{D} = E$ then each element of $\mathfrak{G}$ may be represented uniquely in the form HK and also in the form $K'H'$ (where $H, H' \in \mathfrak{H}$ and $K, K' \in \mathfrak{K}$). Therefore in $HK = K'H'$ the elements H' and K' are uniquely determined for any given H and K. If K is fixed and H runs over all elements of $\mathfrak{H}$, then H' also runs over all elements of $\mathfrak{H}$. The system Σ_K of the corresponding solutions K' of the equation $HK = K'H'$ (each K' taken as many times as it is a solution) is uniquely defined by K.

Let the order of $\mathfrak{H}$ be not less than the order of $\mathfrak{K}$. Then for every K, the system Σ_K contains K at least twice, in fact, if $K = E$, then the statement is evident. If $K \neq E$, then the system Σ_K does not contain the unit element, thus the statement also holds. The elements H and H' for which the relation $HK = KH'$ holds, form two groups $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. Clearly, $K^{-1}\mathfrak{M}K = \mathfrak{M}'$. If all Σ_K contain no elements other than K, then $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$. If $K \neq \overline{K} \in \Sigma_K$ then $\overline{K}^{-1}\mathfrak{M}\overline{K} = \mathfrak{M}'$; in fact, if $K^{-1}\mathfrak{M}K = \mathfrak{M}'$ and $\overline{H}K = \overline{K}\overline{H}'$ then $\overline{H}'^{-1}\overline{K}^{-1}\overline{H}\mathfrak{M}\overline{H}^{-1}\overline{K}\overline{H}' = \mathfrak{M}'$, i. e. $\overline{K}^{-1}\mathfrak{M}\overline{K} = \mathfrak{M}'$ since $\mathfrak{H}$ is an abelian group. For the element $K\overline{K}^{-1} = K^*$ we have $K^{*-1}\mathfrak{M}K^* = \mathfrak{M}$ thus every element of Σ_{K^*} transforms the group $\mathfrak{M}$ into itself.

Therefore if $\{\Sigma_{K^*}\} = \mathfrak{K}$, then $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$.

If $\{\Sigma_{K^*}\} = \mathfrak{K}' \subset \mathfrak{K}$, then the elements $\mathfrak{H}\mathfrak{K}'$ form a proper subgroup of $\mathfrak{G}$. This will proved in two steps:

1) $H_iK^* = K_i^{*\prime}H_i'$ implies $\Sigma_{K^*} = \Sigma_{K_i^{*\prime}}$ $(i = 1, 2, \ldots, h)$. For, if in $H_iK^* = K_i^{*\prime}H_i'$ $(i = 1, 2, \ldots, h$ where h is the order of $\mathfrak{H})$ we substitute K^* by $K^* = H_k^{-1}K_k^{*\prime}H_k'$ (k is fixed) the statement follows.

2) $HK = K'H'$ and $H'\overline{K} = \overline{K}'H''$ $(K, K', \overline{K}, \overline{K}' \in \Sigma_{K^*})$ imply that with $K\overline{K}$ also $K'\overline{K}'$ belongs to $\mathfrak{K}'$.

From 1) and 2) it follows that $\mathfrak{H}\mathfrak{K}' = \mathfrak{K}'\mathfrak{H}$, i. e. the product of $\mathfrak{H}$ and $\mathfrak{K}'$ is a subgroup of $\mathfrak{G}$. Therefore $\mathfrak{G}$ is factorisable by $\mathfrak{H}\mathfrak{K}'$ and $\mathfrak{K}$ thus by Theorem A the group $\mathfrak{G}$ is not simple.

Proof of Theorem 2. (By induction.) The statement is known to hold for groups of order $p \cdot q$ with different primes p and q. Let the order of $\mathfrak{G}$ be n and suppose the theorem true for factorisable groups of order $< n$. Let $\overline{\mathfrak{G}}$ be a proper normal subgroup of $\mathfrak{G}$; the existence of such a subgroup follows from Theorem 1. From Theorem B we may conclude that $\overline{\mathfrak{G}} = \mathfrak{M}\mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are normal subgroups of $\mathfrak{H}$ and $\mathfrak{K}$ respectively. The groups $\mathfrak{H}\mathfrak{M}\mathfrak{N} = \mathfrak{H}\mathfrak{N} = \overline{\mathfrak{H}}$ and $\mathfrak{K}\mathfrak{N}\mathfrak{M} = \mathfrak{K}\mathfrak{M} = \overline{\mathfrak{K}}$ are subgroups of $\mathfrak{G}$ and clearly at least one of them is proper. As is easily seen, $\mathfrak{G}/\mathfrak{M}\mathfrak{N} = \overline{\mathfrak{H}}/\mathfrak{M}\mathfrak{N} \cdot \overline{\mathfrak{K}}/\mathfrak{M}\mathfrak{N}$ and $\overline{\mathfrak{H}}/\mathfrak{M}\mathfrak{N}$ as well as $\overline{\mathfrak{K}}/\mathfrak{M}\mathfrak{N}$ are abelian groups. The orders of the factor-groups $\overline{\mathfrak{H}}/\mathfrak{M}\mathfrak{N}$ and $\overline{\mathfrak{K}}/\mathfrak{M}\mathfrak{N}$ are relatively prime; consequently, by the induction hypothesis, $\mathfrak{G}/\mathfrak{M}\mathfrak{N}$ has a normal subgroup of prime index p. Therefore the group $\mathfrak{G}$ has a normal subgroup of index p; this normal subgroup is factorisable by two of its proper abelian subgroups of relatively prime orders (Theorem B), or it is abelian and so the proof is completed.

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